Unit 4: ALGEBRAIC LANGUAGE

This unit will show you how to:

- Use letters to represent unknown numbers in algebraic expressions
- Use the rules of algebra to write and manipulate algebraic expressions.
- Simplify algebraic expressions by collecting like terms
- Substitute numbers into a formula after simplification
- Add, subtract and multiply polynomials
- Expand and simplify expressions with brackets
- Factorise by removing a common factor
- Expand brackets using special products

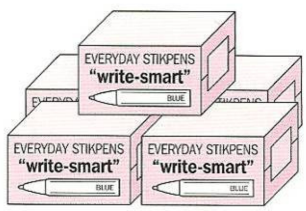
Keywords	
Algebraic expression	Simplify
Monomial	Expand
Coefficient	Factorise
Literal part	Common factor
Degree	Variable
Polynomial	Substitute
Like terms	Evaluate

4.1.- ALGEBRAIC EXPRESSIONS

You can describe everyday situations using algebra.

In **algebra**, you use letters to represent unknown numbers.

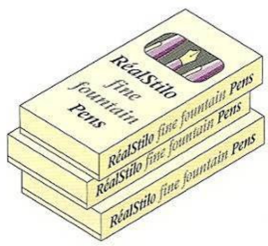
These boxes hold n pens each.



In 5 boxes there are

$n+n+n+n+n=5\times n=5n$ pens

These boxes hold s pens each



In 3 boxes there are 3s pens

There are $5n+3s$ pens in total.

$5n+3s$ is an **algebraic expression**.

An algebraic expression has numbers and letters linked by operations. The letters are called **variables**. Every addend is called **term**.

The expression $5n+3s$ has two terms: **5n** and **3s.**

You can simplify an algebraic expression by collecting **like terms**.

Like terms have exactly the same letters. ($3x^2$ and $-5x^2$ are like terms).

Example: Simplify these expressions:

a) $4x+2y-2x+3y$ b) $7p-3q+5q-p$ c) $5c-2b+2c-3b$

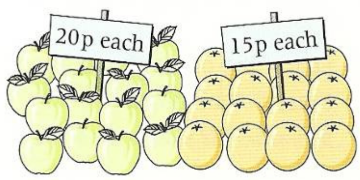
a) $4x+2y-2x+3y=4x-2x+2y+3y=2x+5y$

b) $7p-3q+5q-p=7p-p+5q-3q=6p+2q$

c) $5c-2b+2c-3b=5c+2c-2b-3b=7c-5b$

Example:

In a fruit shop, apples cost 20p each and orange cost 15p each. Write an expression for the cost of x apples and y oranges.



Cost of x apples: $20x$
Cost of y oranges: $15y$

Total cost: $20x+15y$

Exercise 1

In one month, Dan sends *x* texts.

a) Alice sends 4 times as many texts as Dan. How many is this?
b) Kris sends 8 more texts than Alice. How many is this?

Exercise 2

In a pizza takeaway

- a medium pizza has 6 slices of tomato
- a large pizza has 10 slices of tomato

How many slices of tomato are needed for *c* medium pizzas and d large pizzas?

In this unit we will study different types of algebraic expressions: <u>monomials</u>, <u>polynomials</u>, <u>identities</u>, <u>equations</u> and <u>formulae</u>.

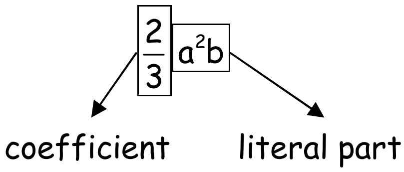
4.2.- MONOMIALS AND POLYNOMIALS

A **monomial** is an algebraic expression containing one term which may be a number, a variable or a product of numbers and variables, with no negative or fractional exponents. (*Mono* implies *one* and the ending *nomial* is Greek for *part*)

For example: $\frac{2}{3}a^2b$, x^2 , $-2xy$, 13 , $520x^2y^4$ are monomials

$\frac{1}{x}$, $x^{\frac{1}{2}}$ are NOT monomials

The number is called **coefficient** and the variables are called **literal part**.



The **degree** is the sum of the exponents of every variable.

Example: the degree of $\frac{2}{3}a^2b$ is $2+1=3$

Exercise 3

For every monomial write the coefficient, the literal part and the degree:

Monomial	$5x$	$0.3x^4$	$-\frac{1}{9}x^2y$	$\frac{xy}{8}$	$-\frac{3x^2y^3}{7}$	$-x^2y^4$
Coefficient						
Literal part						
Degree						

A **polynomial** is an algebraic sum of monomials. (*Poly* implies *many*)

For example: x^2+2x , $3x+x^2+5x+6$, $4x-6y+8$

If there are two monomials, it is called a **binomial**, for example x^2+2x

If there are three monomials, it is called a **trinomial**, for example $4x-6y+8$

The degree of the entire polynomial is the degree of the highest-degree term that it contains.

Examples: $x^4 - 7x^3$ is a fourth-degree binomial

$4x^2 + 2x - 7$ is a second-degree trinomial

terms

$$4x^2 + 2x - 7$$

Leading term Constant term

Polynomials are usually written this way, with the terms written in '**decreasing**' order; that is, with the highest exponent first, the next highest next, and so forth, until you get down to the constant term.

Polynomials are also sometimes named for their degree:

- a second-degree polynomial, such as $4x^2$, $x^2 - 9$, or $ax^2 + bx + c$, is also called a '**quadratic**'.
- a third-degree polynomial, such as $-6x^3$ or $x^3 - 27$, is also called a '**cubic**'
- a fourth-degree polynomial, such as x^4 or $2x^4 - 3x^2 + 9$, is sometimes called a '**quartic**'.

Evaluating a polynomial is the same as calculating its number value at a given value of the variable. For instance:

Evaluate $2x^3 - x^2 - 4x + 2$ at $x = -3$

$$2(-3)^3 - (-3)^2 - 4(-3) + 2 = 2(-27) - 9 + 12 + 2 = -54 - 9 + 12 + 2 = -63 + 14 = \underline{\underline{-49}}$$

Exercise 4

Evaluate the polynomial $x^3 - 2x^2 + 3x - 4$ at the given values of x:

$x = 0$	
$x = 1$	
$x = 2$	
$x = -1$	

Adding and subtracting polynomials

You can add or subtract monomials only if they have the same literal part, that is, if they are **like terms**. In this case, you sum or subtract the coefficients and leave the same literal part.

Look at these examples:

$$4xy^2 + 3xy^2 = 7xy^2 \qquad 5x^2 + 3 - 2x^2 - 1 = 3x^2 + 2$$

There are two ways of adding or subtracting polynomials:

$$A = 3x^3 + 3x^2 - 4x + 5 \qquad B = x^3 - 2x^2 + x - 4$$

$A+B$:

Horizontally:

$$A + B = \left(3x^3 + 3x^2 - 4x + 5\right) + \left(x^3 - 2x^2 + x - 4\right) =$$
$$= 3x^3 + 3x^2 - 4x + 5 + x^3 - 2x^2 + x - 4 = \underline{\underline{4x^3 + x^2 - 3x + 1}}$$

Vertically:

$$\begin{array}{rl} A = & 3x^3 + 3x^2 - 4x + 5 \\ B = & x^3 - 2x^2 + x - 4 \\ \hline A + B = & 4x^3 + x^2 - 3x + 1 \end{array}$$

$A-B$:

Horizontally:

$$A - B = \left(3x^3 + 3x^2 - 4x + 5\right) - \left(x^3 - 2x^2 + x - 4\right) =$$
$$= 3x^3 + 3x^2 - 4x + 5 - x^3 + 2x^2 - x + 4 = \underline{\underline{2x^3 + 5x^2 - 5x + 9}}$$

Vertically: notice that $A - B = A + \left(-B\right)$

$$\begin{array}{rl} A = & 3x^3 + 3x^2 - 4x + 5 \\ -B = & -x^3 + 2x^2 - x + 4 \\ \hline A - B = & 2x^3 + 5x^2 - 5x + 9 \end{array}$$

Exercise 5

Do the following calculations with polynomials:

a) $\left(2x^2 + 5x\right) + \left(x^3 - 2x\right)$

b) $\left(x^2 + 2x\right) - \left(x^2 + 2x\right)$

c) $\left(2x^3 - 2\right) + \left(3x^4 - 2x\right)$

d) $\left(2x^3 - x^2 + 3x - 1\right) - \left(x^3 + 5x^2 - 3x + 4\right)$

Multiplying monomials and polynomials

- Monomial x polynomial: distribute the monomial through the brackets.

  For example:

  $$-3x\left(4x^2-x+10\right)=-3x\cdot 4x^2-3x\cdot(-x)-3x\cdot 10=\underline{\underline{-12x^3+3x^2-30x}}$$

  Multiply all the terms inside the bracket by the term outside is called **expanding the bracket**.

- Polynomial x polynomial. Look at these examples:

  - Simplify $(x+3)(x+2)$

  The first way you can do this is 'horizontally', where you distributive twice:

  $$(x+3)(x+2)=x^2+2x+3x+6=\underline{\underline{x^2+5x+6}}$$

  In this case, we **expand double brackets**, that is, each term in the first bracket multiplies each term in the second bracket.

  The 'vertical' method is much simpler, because it is similar to the multiplications learnt at the primary school:

  - Simplify $\left(4x^2-4x-7\right)(x+3)$

$$\begin{array}{rrrr} & 4x^2 & -4x & -7 \\ & & x & +3 \\ \hline & -12x^2 & -12x & -21 \\ 4x^3 & -4x^2 & -7x & \\ \hline 4x^3 & -8x^2 & -19x & -21 \end{array}$$

Exercise 6

Expand and simplify:

a) $\left(2x^2+3\right)(x-1)-x(x-2)$

b) $(x+4)\left(2x^2+3x-5\right)-3x(-x+1)$

c) $\left(x^2-5x+3\right)\left(x^2-x\right)-x\left(x^3-3\right)$

d) $\left(\frac{1}{2}x^2+\frac{5}{3}x+\frac{1}{6}\right)(6x-12)$

Exercise 7

A small box contains 12 chocolates. Sam buys y small boxes of chocolates.

a) Write an expression for the number of chocolates Sam buys.

A large box contains 20 chocolates. Sam buys 2 more of the large boxes than the small ones.

b) Write an expression for the number of the large boxes of chocolates he buys.
c) Find, in terms of y, the total number of chocolates in the large boxes that Sam buys.
d) Find, in terms of y, the total number of chocolates Sam buys. Give your answer in its simplest form.

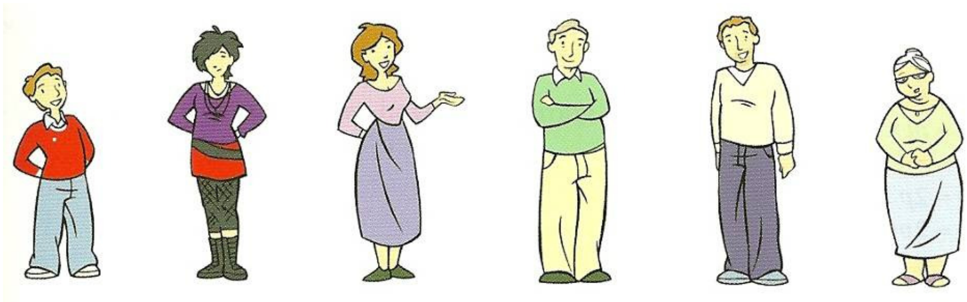
Exercise 8

Jake is n years old.
Jake's sister is 4 years older than Jake.
Jake's mother is 3 times older than his sister.
Jake's father is 4 times older than Jake.
Jake's uncle is 2 years younger than Jake's father.
Jake's grandmother is twice as old as Jake's uncle.

a) Copy the table and write each person's age in terms of n.

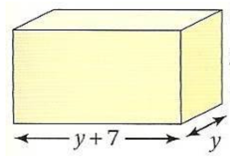
Jake	Sister	Mother	Father	Uncle	Grandmother
n					

b) Find, in terms of n, how much older Jake's grandmother is than his mother. Give your answer in its simplest form.



Exercise 9

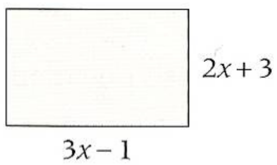
Write an expression involving brackets for the volume of this cuboid.



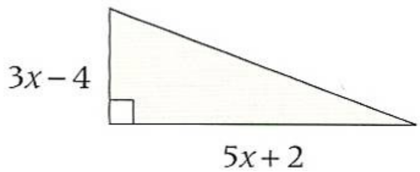
Expand the brackets and simplify your expression.

Exercise 10

a) The diagram shows a rectangle with an area of 75 cm^2. Show that $6x^2+7x-78=0$.



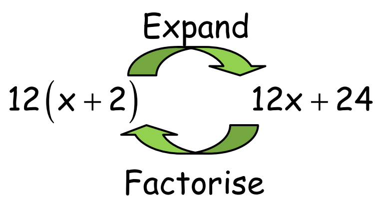
b) This triangle also has an area of 75 cm^2. Show that $15x^2=14x+158$.



Factorising

The reverse of expanding a set of brackets is called **factorising**.

To factorise an expression, you put brackets in.



To factorise an expression, look for a **common factor** for all the terms.

$3x+9=3(x+3)$ (Write the common factor outside the bracket)

$a^2-a=a(a-1)$

$(p+q)^2-2(p+q)=(p+q)((p+q)-2)=(p+q)(p+q-2)$

You can check your answer by expanding.

Exercise 11

The cards show expansions and factorisations. Match the cards in pairs.

$4(x+3)$	$4x^2-3x$	$3(x-4)$	$4x+3x^2$
$3x-12$	$x(4+3x)$	$4x+12$	$x(4x-3)$

Exercise 12

Factorise these expressions.

a) $5x^2-15x^3+25x^4$

b) $\frac{x^4}{3}-\frac{x}{9}-\frac{1}{15}$

c) $2x^3y^5-3x^2y^4+2x^7y^2+7x^3y^3$

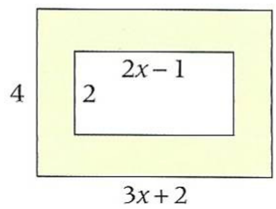
d) $2(x-3)+3(x-3)-5(x-3)$

e) $2xy^2-6x^2y^3+4xy^3$

f) $\frac{(x^2-3)}{2}(y-1)-\frac{7}{2}(y-1)$

Exercise 13

Show that the shaded area of this rectangle is $2(4x+5)$.



4.3.- IDENTITIES, FORMULAE AND EQUATIONS

An **identity** is true for all values of x. For example:

$x(x+1)\equiv x^2+x$ Whatever value of x you try, this statement is always true.

$\equiv$ means 'is identical to'

An **equation** is only true for a limited number of values of x. For example:

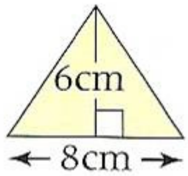
$2x+1=5$ is only true when $x=2$.

A **formula** describes the relationship between two or more variables.

For example: the formula for the area of a triangle is $A = \frac{1}{2}bh$

You can **substitute** numbers into a formula to work out the value of a variable.

For example:



$$A = \frac{1}{2}bh = \frac{1}{2} \times 8 \times 6 = 24\ \text{cm}^2$$

Example: Decide if each of these statements is an identity, an equation or a formula.

a) $x^3 - 2x = x(x^2 - 2)$ b) $5x - 1 = 2x + 3$ c) $A = \pi r^2$

a) Expand the right-hand side: $x(x^2 - 2) = x^3 - 2x$

$x^3 - 2x =$ left-hand side, so the statement is an identity.

b) $5x - 1 = 2x + 5$

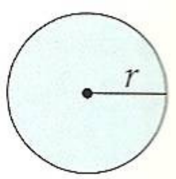
$5x - 1 = 2x + 5$ This statement is only true for one value of x.

$3x = 6$

$x = 2$

$5x - 1 = 2x + 5$ is an equation.

c) $A = \pi r^2$ is a formula showing the relationship between the radius and the area of a circle.



If you know the value of r, you can find A from the formula and vice versa.

Exercise 14

Copy these statements and say whether they are identities, equations or formulae.

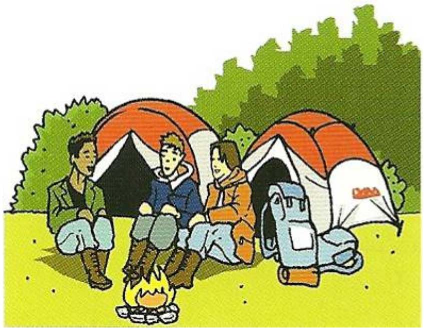
a) $c = 2\pi r$ b) $3x(x + 1) = 3x^2 + x$ c) $3x + 1 = 10$

d) $y \cdot y = y^2$ e) $2x + 5 = 3 - 7x$ f) $A = \frac{1}{2}(a + b)h$

g) $a^2 = b^2 + c^2$ h) $20 - x = -(x - 20)$ i) $2x^2 = 50$

Exercise 15

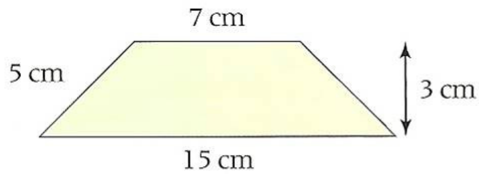
A campsite charges for the pitching of a tent and the number of the people, p, that stay in it. If C is the total cost in pounds, the formula used by the campsite is $C = 2p + 5$



a) Work out the cost of pitching a tent for 6 people.
b) If the cost is £23, how many people are sleeping in tent?
c) Explain why the cost could never be £26.

Exercise 16

a) The formula for the area of a trapezium is $A = \frac{1}{2}(a+b)h$, where a and b represent the lengths of the parallel sides an h is the perpendicular height. Find the area of this trapezium.

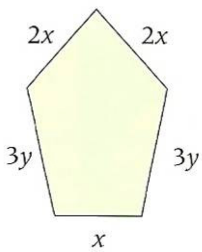


b) If another trapezium with the same perpendicular height has area 15 cm^2, suggest as many possibilities for the lengths of the parallel sides as you can.

Writing formulae

You can **derive a formula** from information you are given.

Example: Derive a formula for the perimeter of this pentagon.



Let P represent the perimeter.

$P = 2x + 2x + 3y + x + 3y$

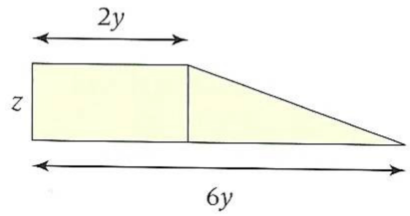
$P = 5x + 6y$

Exercise 17

Write a formula to show you total amount of pocket money P (£s), if you receive £3 per month with an extra £2 for every job (j) you do at home.

Exercise 18

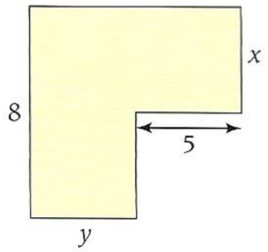
Write a formula for the total area of this shape. Let the total area be A.



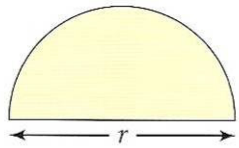
Exercise 19

Write your own formula to represent these quantities.

a) The perimeter, P, of this hexagon.



b) The area, A, of a semicircle.

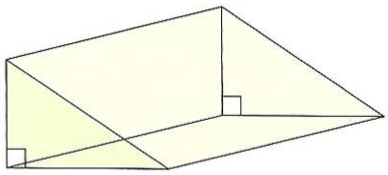


Exercise 20

Megan is writing a formula to work out the volume of this prism.

Her formula is

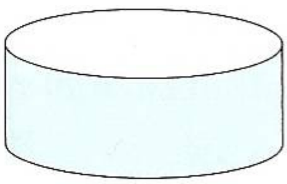
$$V = \frac{abc}{2}$$



Explain what a, b and c stand for.
Explain why Megan's formula is correct.

Exercise 21

This cylinder has base radius r and height h. Write a formula for V, the volume of the cylinder.



Powers and special products

Some products occur so frequently in algebra that is advantageous to be able to recognize them by sight. This will be particularly useful when we talk about factorising. This year we are going to learn three kinds of special products.

<u>The square of a sum</u>

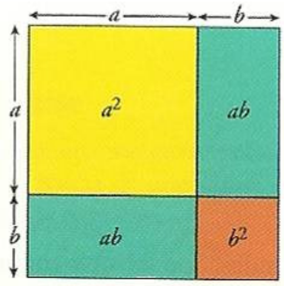
$$(a+b)^2=(a+b)(a+b)=aa+ab+ba+bb=\underline{\underline{a^2+2ab+b^2}}$$

The rule in words:

'The square of the sum of two terms is the sum of their squares <u>plus</u> two times the product of the terms'.

$$\boxed{(a+b)^2=a^2+2ab+b^2}$$

The picture below shows a 'geometric' proof of the square of the sum:



Examples:

$$(x+2)^2=x^2+2\cdot x\cdot 2+2^2=\underline{\underline{x^2+4x+4}}$$
$$(2x+3)^2=(2x)^2+2\cdot 2x\cdot 3+3^2=\underline{\underline{4x^2+12x+9}}$$

Be careful here: 'the square of the sum' and 'the sum of the squares' sound very similar, but are different; the square of the sum is $(a+b)^2$ and the sum of squares is a^2+b^2.

$$(3+2)^2\neq 3^2+2^2$$

Exercise 22

Expand as in the examples:

a) $(3x+2)^2$ b) $\left(\frac{x}{2}+\frac{3}{4}\right)^2$ c) $(ax+b)^2$

Exercise 23

Compute 101^2 without pencil and paper.

The square of a difference

$$(a-b)^2 = (a+(-b))^2 = a^2 + 2 \cdot a \cdot (-b) + (-b)^2 = \underline{\underline{a^2 - 2ab + b^2}}$$

The rule in words:

'The square of the difference of two terms is the sum of their squares minus two times the product of the terms'.

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

Examples:

$$(x-5)^2 = x^2 - 2 \cdot x \cdot 5 + 5^2 = \underline{\underline{x^2 - 10x + 25}}$$

$$(1-2x)^2 = 1^2 - 2 \cdot 1 \cdot 2x + (2x)^2 = \underline{\underline{1 - 4x + 4x^2}}$$

Exercise 24

Expand as in the examples:

a) $(3x-5)^2$ b) $\left(2x^2 - \frac{1}{2}\right)^2$ c) $(4x-y)^2$

Exercise 25

Compute 99^2 without pencil and paper.

The product of a sum and a difference

$$(a+b)(a-b) = a^2 - ab + ba - b^2 = \underline{\underline{a^2 - b^2}}$$

The rule in words:

'The product of a sum and a difference of two terms is the difference of their squares'.

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

Examples:

$$(x+2)(x-2)=x^2-2^2=\underline{\underline{x^2-4}}$$

$$(2x+3)(2x-3)=(2x)^2-3^2=\underline{\underline{4x^2-9}}$$

Exercise 26

Expand as in the examples:

a) $(a+1)(a-1)$ b) $\left(\frac{x}{3}-\frac{1}{2}\right)\left(\frac{x}{3}+\frac{1}{2}\right)$ c) $(3x^2+y)(3x^2-y)$

Exercise 27

Compute 101×99 without pencil and paper.

These three formulas –the square of a sum, the square of a difference, and the difference of squares- are called **short multiplication formulas**.

You should be able to recognize these products both ways. That is, if you see the left side you should think of the right side, and if you see the right side you should think of the left side.

$$(a+b)^2=a^2+2ab+b^2 \qquad a^2+2ab+b^2=(a+b)^2$$

$$(a-b)^2=a^2-2ab+b^2 \qquad a^2-2ab+b^2=(a-b)^2$$

$$(a+b)(a-b)=a^2-b^2 \qquad a^2-b^2=(a+b)(a-b)$$

This is useful when we want to factorise some algebraic expressions or when we want to express some algebraic fractions in its lowest terms. For example:

- $x^2-6x+9=x^2-2\cdot x\cdot 3+3^2=(x-3)^2$
- $x^2-9=x^2-3^2=(x+3)(x-3)$
- $x^2+12x+36=x^2+2\cdot x\cdot 6+6^2=(x+6)^2$
- $\dfrac{x^2-6x+9}{x^2-9}=\dfrac{(x-3)^2}{(x+3)(x-3)}=\dfrac{(x-3)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}}=\dfrac{x-3}{x+3}$

Exercise 28

Factorise these expressions:

a) $x^2 - 25$

b) $x^2 - 18x + 81$

c) $x^2 + 22x + 121$

d) $9 - 16x^2$

e) $x^2 - 2$

f) $100 + 180z + 81z^2$

g) $9x^2 - 12x + 4$

h) $25 - 10y + y^2$

i) $\frac{x^2}{4} + x + 1$

j) $a^2 - \frac{1}{9}$

Exercise 29

Simplify the following expressions:

a) $(x-2)(x+2) - (x^2+4)$

b) $(3x-1)^2 - (3x+1)^2$

c) $(2x-4)^2 - (2x+4)(2x-4)$

d) $(5x-4)(2x+3) - 5$

e) $2(x-5)^2 - (2x^2+3x+50)$

f) $3x^2 - 2(x+5) - (x+3)^2 + 19$

g) $(x+3)^2 - \left[x^2 + (x-3)^2\right]$

Exercise 30

Two integers differ by 2. Multiply them and add 1 to the product. Prove that the result is a perfect square (the square of an integer).