Unit 5: EQUATIONS

This unit will show you how to:

- Solve linear equations after simplifying them
- Solve quadratic equations
- Use trial and improvement to find approximate solutions to equations
- Write equations in order to solve word problems

Keywords	
Variable	Quadratic equation
Unknown	Quadratic formula
Solution	Discriminant
Trial and improvement	Roots
Linear equation	Zeros

5.1.- EQUATIONS. SOLUTION OF AN EQUATION

An **equation** is a statement using algebra that contains an **unknown** quantity and an equals sign.

Examples: $2x+3=15$, $x^2-16=0$, $3^x=81$, $\frac{2}{x}=5$, $\sqrt{x+1}=5$, $x+y=5$

The **solution** of an equation is the set of values which, when substituted for unknowns, make the equation a true statement.

Example: There are two values that make true the following equation:

$$x^2-16=0 \qquad \text{Solution: } \{x=-4,\ x=4\}$$

There are different types of equations. For example:

- **Polynomial equations.** $x^2-2=\frac{x+1}{3}$; $x^3-9x=(x+1)^2+3$
- **Surd equations.** $\sqrt{x+17}+2=x-1$
- **Exponential equations.** $2^x=64$
- **Rational equations.** $\frac{x+2}{x+3}-\frac{3}{x-1}=\frac{1}{8}$

Exercise 1

Are the values $x=3$ or $x=-2$ solution of any of the following equations?

a) $\frac{3-x}{5}+\frac{x}{3}=\frac{1}{3}$ b) $2^x+2^{x-1}-2^{x+1}=-4$

c) $(2-x)^3+3x=x^2-1$ d) $\sqrt{14-x}=4$

Degree of an equation

The **degree of an equation** that has not more than one variable in each term is the exponent of the highest power to which that variable is raised in the equation.

The equation $3x-17=0$ is a **first-degree equation**, since x is raised only to the first power.

An example of a **second-degree equation** is $5x^2-2x+1=0$.

An example of a **third-degree equation** is $4x^3-7x^2=0$.

The equation $3x-2y=5$ is of the first degree in two variables, x and y. When more than one variable appears in a term, as in $xy=5$, it is necessary to add the exponents of the variables within a term to get the degree of the equation. Since $1+1=2$, the equation $xy=5$ is of the second degree.

Trial and improvement

In this unit we will study the polynomial equations of degrees 1 and 2 (in one variable), that is, linear equations and quadratic equations.

However, you can use **trial and improvement** to estimate the solutions of any equation.

- You **estimate** a solution and **try** it in the equation.
- If your estimate doesn't fit, you **improve** it and try again.

Example:

$x^3+x=33$ has a solution between 3 and 4.
Use trial and improvement to find the solution.
Give your answer correct to 1 decimal place.

x	x^3+x	Too big or too small?
3.5	46.375	too big
3.2	35.968	too big
3.1	32.891	too small
3.15	34.406	too big

The solution is between 3.1 and 3.15. All the values between 3.1 and 3.15 round to 3.1.

The solution is 3.1 to 1 dp.

Exercise 2

Use trial and improvement to find a solution to the following equations:

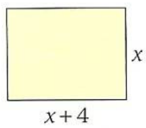
a) $5^x = 20$ b) $x^4 - x^2 = 54$

Give your answers to 2 decimal places.

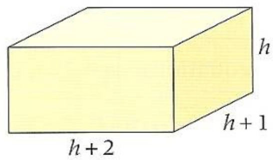
Exercise 3

Write an equation to solve each problem. Use trial and improvement to find the exact answer.

a) The area of this rectangle is 77 cm^2.
What are the length and width?



b) The volume of this cuboid is 990 cm^3.
What are the dimensions?



5.2.- FIRST-DEGREE EQUATIONS

A **first-degree equation** is called a **linear equation**. The highest exponent of a linear equation is 1. The standard form for a linear equation is:

$ax + b = c$ **a**, **b** and **c** can have any value, except that **a** can't be 0

Examples:

a) $3x + 2 = -10$. The solution of this equation is $x = -4$.

b) $4x - 6 = 4(x+3) \Rightarrow 4x - 6 = 4x + 12 \Rightarrow 0x = 18 \Rightarrow 0 = 18$ False equation.
This equation has no solution.

c) $4x - 6 = 4(x-2) + 2 \Rightarrow 4x - 6 = 4x - 6 \Rightarrow 0x = 0$ This equation is true for any value of x. Any number is solution of this equation.

Steps for solving a first degree equation in one variable

1. If the equation contains fractions, multiply each side of the equation by the least common multiple (LCM) of the denominators to clear the equation of fractions.

2. Use the Distributive Property to remove brackets.

3. Combine any like terms on the right side of the equation and any like terms on the left side of the equation.

4. Use the Addition or Subtraction Property to rewrite the equation with only one variable term and only one constant term.

   Addition Property: if $a = b$, then $a + c = b + c$.
   Subtraction Property: if $a = b$, then $a - c = b - c$.

   Example:

   $$x + 3 = -2$$
   $$x + 3 - 3 = -2 - 3$$
   $$x = -2 - 3$$
   $$x = -5$$

   Tip: we "leave out" the first step and we say: terms can "move" from the left-hand side to the right-hand side (and vice versa) by changing their sign.

   $$x + 3 = -2$$
   $$x = -2 - 3$$
   $$x = -5$$

5. Use the Multiplication or Division property to rewrite the equation with the variable alone on one side of the equation and a constant term on the other side of the equation.

   Multiplication Property: if $a = b$ and $c \neq 0$, then $a \cdot c = b \cdot c$.

   Division Property: if $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

   Example:

   $$3x = 15$$
   $$\frac{3x}{3} = \frac{15}{3}$$
   $$x = 5$$

   Tip: we "leave out" the first step and we say: the number that is being multiplied on one side "is moved" to the other side and its terms are divided by it (and vice versa).

   $$3x = 15$$
   $$x = \frac{15}{3}$$
   $$x = 5$$

6. Check the solution.

Example:

$$\frac{3x-1}{20}-\frac{2(x+3)}{5}=\frac{4x+2}{15}-5$$

$$LCM(20,5,15)=60$$

$$60\cdot\left(\frac{3x-1}{20}-\frac{2(x+3)}{5}\right)=60.\left(\frac{4x+2}{15}-5\right)$$

$$3(3x-1)-24(x+3)=4(4x+2)-300$$

$$9x-3-24x-72=16x+8-300$$

$$9x-24x-16x=8-300+3+72$$

$$-31x=-217$$

$$x=\frac{-217}{-31}=7$$

Solution: $\boxed{x=7}$

We can check the solution replacing x with the number 7 in both sides of the equation:

$$\left.\begin{array}{ll}\text{Left hand side:} & \dfrac{3\cdot 7-1}{20}-\dfrac{2(7+3)}{5}=-3\\ \text{Right hand side:} & \dfrac{4\cdot 7+2}{15}-5=-3\end{array}\right\}\; -3=-3 \text{, so our solution is correct.}$$

Exercise 4

Solve the following equations:

a) $3x-\frac{x+3}{4}=13$

b) $2(x-3)+1=3(x-1)-(2+x)$

c) $4(2x+1)-3(x+3)=5(x-2)$

d) $\frac{x}{2}-\frac{2(x+2)}{7}=\frac{x-3}{4}$

e) $\frac{x-4}{8}+\frac{9-x}{12}-\frac{2x-7}{24}+5=x-8$

f) $\frac{(1+x)^2}{5}=\frac{2x+4}{25}+\frac{x^2}{5}+\frac{1}{5}$

g) $\frac{(2x-1)(2x+1)}{4}=\frac{3(4x^2+1)}{12}-x$

h) $(x-3)(x+3)=\frac{3(x-1)}{2}+x^2$

5.2.- QUADRATIC EQUATIONS

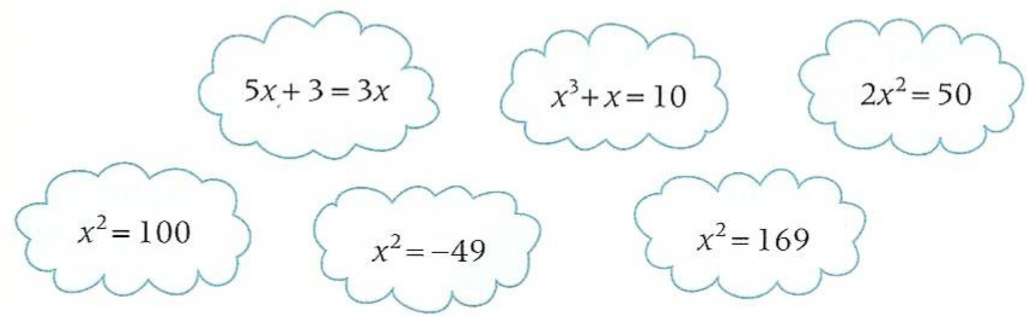
A **second-degree equation** is called a **quadratic equation**. The highest exponent of a quadratic equation is 2. The standard form for a quadratic equation is:

$ax^2 + bx + c = 0$ **a**, **b** and **c** can have any value, except that **a** can't be 0

Examples: $5x^2 - 3x + 3 = 0$, $x^2 - 4 = 0$, $x^2 + 5x = 0$

Exercise 5

Here are six equations.



a) Write the four quadratic equations.

b) From the quadratic equations, find three that have two solutions. Write these three equations and their solutions.

How to solve quadratic equations?

Many quadratic equations can be solved by **factorising**.

For example, solve the equation $x^2 + 5x + 6 = 0$.

$x^2 + 5x + 6 = 0$	The two factors will each start with x.
$(x+\ \ )(x+\ \ ) = 0$	Now find two numbers with a **sum** of 5 (for 5x) and a **product** of 6.
$(x+3)(x+2) = 0$	Check; $(x+3)(x+2) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6$

The two factors have **product** of zero. This means that at least one of them is equal to zero.

Either $x + 3 = 0$ or $x + 2 = 0$

If $x + 3 = 0$, $x = -3$ and if $x + 2 = 0$, $x = -2$

The solutions are $x = -3$ and $x = -2$.

The key to solving quadratic equations is that the product of the two factor is zero, so you can say that one or other (or both) of the factors is zero. This leads to the two solutions.

Exercise 6

Solve these quadratic equations by factorising them into double brackets.

a) $x^2+7x+12=0$ b) $x^2+8x+12=0$ c) $x^2+10x+25=0$

d) $x^2+5x-14=0$ e) $x^2-4x-5=0$ f) $x^2-5x+6=0$

g) $x^2-10x+21=0$ h) $x^2=3x+40$ i) $x^2-6x+9=0$

Exercise 7

Explain why $x^2+7x+11=0$ cannot be solved by factorisation.

We have solved quadratic equations by factorising, but how to solve **any** quadratic equation?

Quadratic equations can be solved using a special formula called the **quadratic formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The $\pm$ means you need to do a **plus** AND a **minus**, and so there are normally TWO solutions!

The answers it gives are the solutions to the quadratic equation, and are often called **roots**, or sometimes **zeros**.

Examples:

a) $x^2-6x+5=0$ $a=1,\quad b=-6,\quad c=5$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{6\pm\sqrt{(-6)^2-4\cdot1\cdot5}}{2\cdot1}=\frac{6\pm\sqrt{36-20}}{2}=$$

$$=\frac{6\pm\sqrt{16}}{2}=\frac{6\pm4}{2}\quad\begin{cases}\dfrac{6+4}{2}=5\\[2ex]\dfrac{6-4}{2}=1\end{cases}$$

This equation has **two different solutions**: $x_1=5,\quad x_2=1$.

b) $4x^2+4x+1=0$ $\quad a=4,\ \ b=4,\ \ c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{4^2-4\cdot4\cdot1}}{2\cdot4}=\frac{-4\pm\sqrt{16-16}}{8}=$$

$$=\frac{-4\pm\sqrt{0}}{8}=\frac{-4\pm0}{8}\quad\begin{cases}\dfrac{-4+0}{8}=-\dfrac{1}{2}\\[2ex]\dfrac{-4-0}{8}=-\dfrac{1}{2}\end{cases}$$

Both solutions are equal. We say that this equation has a **double solution** or **double root**.

c) $x^2-2x+5=0$ $\quad a=1,\ \ b=-2,\ \ c=5$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{(-2)^2-4\cdot1\cdot5}}{2\cdot1}=\frac{2\pm\sqrt{4-20}}{2}=$$

$$=\frac{2\pm\sqrt{-16}}{2}\quad\begin{cases}\dfrac{2+\sqrt{-16}}{2}\ \text{is not a real number}\\[2ex]\dfrac{2-\sqrt{-16}}{2}\ \text{is not a real number}\end{cases}$$

Up to now, you just know rational and irrational numbers. These numbers, all together, are called **real numbers**. We use the letter $\mathbb{R}$ to refer to them. So, the previous equation has no solution into the set of real numbers.

$\dfrac{2+\sqrt{-16}}{2}$ and $\dfrac{2-\sqrt{-16}}{2}$ are **complex numbers**.

Let $\sqrt{-1}$ be i, the **imaginary unit** $\left(i=\sqrt{-1}\right)$.

$$\sqrt{-16}=\sqrt{16\cdot(-1)}=\sqrt{16}\cdot\sqrt{-1}=4i$$

$$\frac{2+\sqrt{-16}}{2}=\frac{2+4i}{2}=\frac{2}{2}+\frac{4i}{2}=1+2i \qquad \frac{2-\sqrt{-16}}{2}=\frac{2-4i}{2}=\frac{2}{2}-\frac{4i}{2}=1-2i$$

Our equation has **two complex solutions**: $x_1=1+2i,\quad x_2=1-2i$

The previous examples show that the different types of solutions of the second-degree equations depend on the value of b^2-4ac.

b^2-4ac is called the **discriminant**.

- When b^2-4ac is **positive**, you will get **two different solutions**.
- When b^2-4ac is **zero**, you get **one double solution**.
- When b^2-4ac is **negative**, you get **two complex solutions**.

Sometimes you can find **"incomplete"** quadratic equations, for example:

$$16x^2-25=0\ ,\quad 7x^2+11x=0$$

You can solve these equations in an easy way, without using the quadratic formula:

$$16x^2-25=0\Rightarrow 16x^2=25\Rightarrow x^2=\frac{25}{16}\Rightarrow x=\pm\sqrt{\frac{25}{16}}=\pm\frac{5}{4}$$

$$7x^2+11x=0\Rightarrow x(7x+11)=0\Rightarrow\begin{cases}x=0\\7x+11=0\Rightarrow x=-\dfrac{11}{7}\end{cases}$$

Exercise 8

Solve these quadratic equations.

a) $x^2-5x+6=0$ b) $9x^2+6x+1=0$ c) $5x^2-7x+3=0$

d) $2x^2+5x-3=0$ e) $x^2-3x+15=0$ f) $7x^2-28=0$

g) $11x^2-37x=0$ h) $7x^2+5=68$ i) $2x^2+98=0$

5.3.- SOLVING PROBLEMS USING EQUATIONS

Word problems are a series of expressions that fits into an equation.

For example:

Two cyclists start at the same time from opposite ends of a course that is 45 miles long. One cyclist is riding at 14 mph and the second cyclist is riding at 16 mph. How long after they begin will they meet?

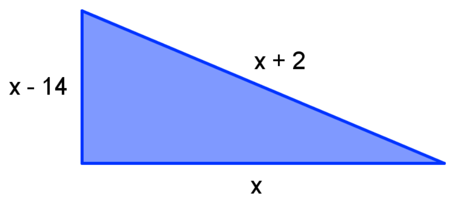
Steps to solve math word problems:

1. Read the problem entirely (get a feel for the whole problem).
2. List information and the variables you identify (attach units of measure to the variables).

3. Define what answer you need, as well as its units of measure.
4. Translate the wording into a numeric equation.
5. Solve the equation.
6. Check your solution using the problem (not your equation).

Examples:

1) In a right triangle, one leg is 2 cm less than the hypotenuse and 14 cm more than the other leg. Find out the length of the three sides.



The Pythagorean Theorem links the three sides of the triangle:

$$(x-14)^2 + x^2 = (x+2)^2$$

$$x^2 - 28x + 196 + x^2 = x^2 + 4x + 4$$

$$x^2 - 32x + 192 = 0$$

$$x = \frac{32 \pm \sqrt{32^2 - 4\cdot 1\cdot 192}}{2\cdot 1} = \frac{32 \pm \sqrt{256}}{2} = \frac{32 \pm 16}{2} \quad \begin{cases} \frac{32+16}{2} = 24 \\ \frac{32-16}{2} = 8 \end{cases}$$

If $x = 24 \Rightarrow x - 14 = 10,\ x + 2 = 26$.

If $x = 8 \Rightarrow x - 14 = -6,\ x + 2 = 10$.

Since the length of one side can't be a negative number, the only valid solution will be: 10, 24 and 26 cm.

2) A pastrycook mixed 12 kg of sugar that costs 1.10 €/kg with a certain quantity of honey that costs 4.20 €/kg. The price of the mixture is 2.34 €/kg. How many kg of honey did he use?

	kg	Price (€/kg)	Cost
Sugar	12	1.10	$1.10\cdot 12 = 13.20$
Honey	x	4.20	$4.20x$
Mixture	$12 + x$	2.34	$2.34(12+x)$

From the last column, you get the equation: $13.20 + 4.20x = 2.34(12+x)$.

Let's solve for x.

$$13.20 + 4.20x = 2.34(12 + x) \Rightarrow 13.20 + 4.20x = 28.08 + 2.34x$$

$$4.20x - 2.34x = 28.8 - 13.20 \Rightarrow 1.86x = 14.88 \Rightarrow x = \frac{14.88}{1.86} = 8$$

Solution: 8 kg of honey.

Exercise 9

The sum of three consecutive odd integers is 117. Find the numbers.

Exercise 10

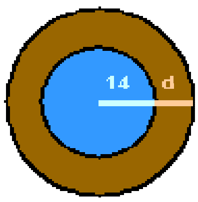
If the height of a triangle is 5 cm less than the length of its base, and if the area of the triangle is 52 cm^2, find the base and the height.

Exercise 11

The smallest angle of a triangle is two-thirds the size of the middle angle, and the middle angle is three-sevenths of the largest angle. Find all three angle measures.

Exercise 12

A circular swimming pool with a diameter of 28 feet has a deck of uniform width built around it. If the area of the deck is 60π square feet, find its width. (1 foot = 30.48 cm)



Exercise 13

If one side of a square is doubled in length and the adjacent side is decreased by 2 cm, the area of the resulting rectangle is 96 cm^2 larger than that of the original square. Find the dimensions of the rectangle.

Exercise 14

If the sum of the legs of a right triangle is 49 cm and the hypotenuse is 41 cm, find the two legs.

Exercise 15

The price of a pair of shoes was increased by 15% in December and was decreased by 20% in January. And so then, the original price has decreased 6.96 €. Find the original price of the shoes.

Exercise 16

Find the selling price per pound of a coffee mixture made from 8 pounds of coffee that sells for $9.20 per pound and 12 pounds of coffee that costs $5.50 per pound. (1 pound = 454 g)

Exercise 17

In three more years, Miguel's grandfather will be six times as old as Miguel was last year. When Miguel's present age is added to his grandfather's present age, the total is 68. How old is each one now?

Exercise 18

Two cyclists start at the same time from opposite ends of a course that is 45 miles long. One cyclist is riding at 14 mph and the second cyclist is riding at 16 mph. How long after they begin will they meet? (1 mile = 1.609 km)

Exercise 19

A car and a bus set out at 2 p.m. from the same point, headed in the same direction. The average speed of the car is 30 mph slower than twice the speed of the bus. In two hours, the car is 20 miles ahead of the bus. Find the rate of the car.

Exercise 20

Suppose one painter can paint the entire house in twelve hours, and the second painter takes eight hours. How long would it take the two painters together to paint the house?