Unit 3: ARITHMETIC AND GEOMETRIC SEQUENCES

This unit will show you how to:

- Find and describe the nth term of a sequence, using this to find other terms
- Describe and find the general term of arithmetic and geometric sequences
- Find the sum of n terms in arithmetic and geometric sequences
- Use sequences to solve word problems

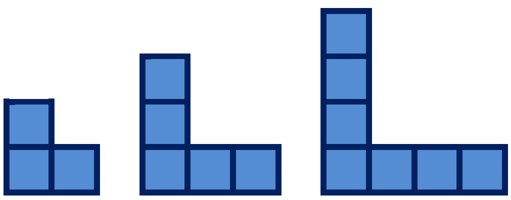
Keywords	
Sequence	Arithmetic sequence
Term	Geometric sequence
Rule	Common difference
General term	Common ratio
Nth term	

3.1.- SEQUENCES

A **sequence** is a set of numbers that follow a pattern, for example:

- 5, 9, 13, 17, 21, … are the first five **terms** of a sequence that goes up in 4s.
- 3, 6, 12, 24, 48, … are the first five terms of a sequence that doubles.
- 1, 4, 9, 16, 25, … is the sequence of square numbers.
- 1, 8, 27, 64, 125, … are the cube numbers

You can find sequences in patterns, for example,



Number of tiles is: 3, 5, 7, …
The next diagram would need 9 tiles.

Example: Write the next two terms in each sequence.

a) 4, 5, 7, 10, 14, 19, … b) 0.6, 0.7, 0.8, 0.9, …

a) 4, 5, 7, 10, 14, 19, … The terms increase by 1, then 2, then 3. So the next two terms are 19 + 6 = 25 and 25 + 7 =32.

b) 0.6, 0.7, 0.8, 0.9, … The terms increase by 0.1. So the next two terms are 0.9 + 0.1 = 1.0 and 1.0 + 0.1 = 1.1.

Exercise 1

Copy each sequence and add the next two terms.

a) 4, 9, 14, 19, 24, ____ , ____ b) 100, 93, 86, 79, 72, ____ , ____
c) 54, 27, 13.5, 6.75, ____ , ____ d) 1, 1, 2, 3, 5, 8, ____ , ____

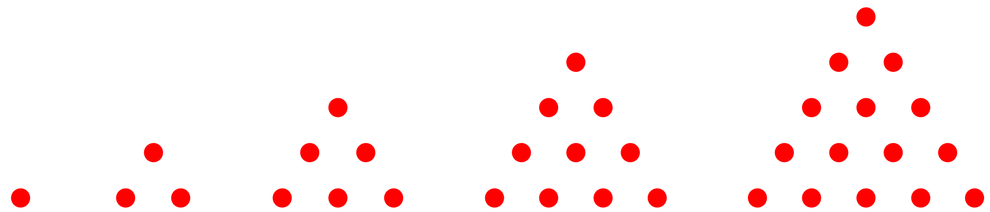
Exercise 2

Write the first five terms of each of these well-known number patterns.

a) Multiples of 3
b) Powers of 2
c) Prime numbers
d) Square numbers over 100

Exercise 3

The triangular numbers form a sequence.



a) Copy the table and use the diagrams to complete it.

Diagrams	1	2	3	4	5
Number of dots					

b) Hence, write the first 10 triangular numbers.
c) Why do you think the square numbers (1, 4, 9, 16, 25, …) got their name?
d) How did the cube numbers get their name? Write the first five cube numbers.

Exercise 4

Look at this number pattern.

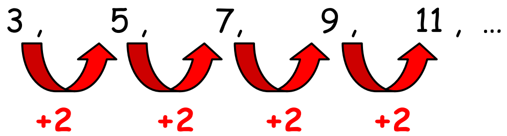
a) Write the next two lines in the pattern.
b) What is $66\ 666\ 666\ 667^2$?
c) What is $\sqrt{444\ 444\ 444\ 888\ 888\ 889}$

$7^2 = 49$
$67^2 = 4489$
$667^2 = 444889$

The general term

A sequence will have a rule that gives you a way to find the value of each term.

Example: the sequence 3, 5, 7, 9, … starts at 3 and jumps 2 every time:



But the rule should be a formula!

Saying "starts at 3 and jumps 2 every time" doesn't tell us how to calculate the:

- 10^{th} term,
- 100^{th} term, or
- n^{th} term, (where **n** could be any term number we want).

We want a formula with "n" in it (where n can be any term number).

So, what would the **Rule** for the sequence 3, 5, 7, 9, … be?

Firstly, we can see the sequence goes up 2 every time, so we can guess that the Rule will be something like "$2 \times n$" (where "n" is the term number). Let's test it out:

Test Rule: 2n

n	Term	Test Rule
1	3	2**n** = 2 x **1** = **2**
2	5	2**n** = 2 x **2** = **4**
3	7	2**n** = 2 x **3** = **6**

That nearly worked … but that Rule is too low by 1 every time, so let us try changing it to:

Test Rule: 2n + 1

n	Term	Test Rule
1	3	2n + 1 = 2 x 1 + 1 = **3**
2	5	2n + 1 = 2 x 2 + 1 = **5**
3	7	2n + 1 = 2 x 3 + 1 = **7**

That works!

So instead of saying "starts at 3 and jumps 2 every time" we write the rule as

the Rule for 3, 5, 7, 9, ... is: **2n+1**

Now, for example, we can calculate the 100th term: 2 x 100 + 1 = 201

Notation:

To make it easier to write down rules, we often use this special style:

a_n represents the **general term (general rule or nth term)** of the sequence

So to mention the "5th term" you just write: a_5

Example: $a_n = 3n - 2$

To find a_5, the 5th term, put n = 5 in the rule.

$$a_5 = 3 \times 5 - 2 = 13$$

Exercise 5

Write the first five terms of each sequence.

a) $a_n = 7n$ b) $a_n = 8n + 2$ c) $a_n = (-n)^3$

d) $a_n = 10 - 2n$ e) $a_n = n^4$ f) $a_n = (n+1)(n+2)$

Exercise 6

Find a formula for the nth term, a_n, of each sequence.

a) 4, 14, 24, 34, 44, ... b) 50, 49.75, 49.50, 49.25, ...

c) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ d) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

Exercise 7

a) What is the nth term formula for $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...?$

b) Plot this sequence on a graph, with the term number on the x-axis and the term on the y-axis.

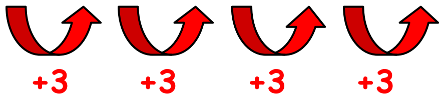
c) What happens if you continue this sequence indefinitely?

3.2.- ARITHMETIC (OR LINEAR SEQUENCES)

If the difference between consecutive terms is the same, the sequence is an **arithmetic or linear sequence**.

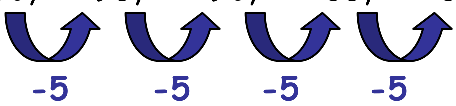
Examples:

a) 12 , 15 , 18 , 21 , 24 , ...



$a_2 - a_1 = 15 - 12 = 3$ $a_3 - a_2 = 18 - 15 = 3$ $a_4 - a_3 = 21 - 18 = 3$

b) 100, 95, 90, 85, 80 , ...



$a_2 - a_1 = 95 - 100 = -5$ $a_3 - a_2 = 90 - 95 = -5$ $a_4 - a_3 = 85 - 90 = -5$

In a linear sequence the terms go up or go down by the same amount.

General term of a linear sequence

Let's find the general term of a linear sequence:

Example: 8, 18, 28, 38, 48, ...

The first term in this sequence is 8 and the difference, **d**, between consecutive terms is 10.

$$a_1 = 8$$
$$a_2 = 8 + 10$$
$$a_3 = 8 + 10 + 10 = 8 + 2 \times 10$$
$$a_4 = 8 + 10 + 10 + 10 = 8 + 3 \times 10$$
$$a_5 = 8 + 10 + 10 + 10 + 10 = 8 + 4 \times 10$$
....
$$a_n = 8 + 10 + 10 + ... + 10 = 8 + (n-1) \times 10 = 8 + 10n - 10 = \underline{\underline{-2 + 10n}}$$

The general term of a linear sequence can be calculated from the first term and the difference using this expression:

$$a_n = a_1 + (n-1)\cdot d$$

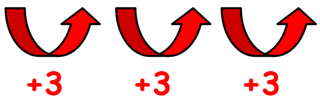
The following expression can also be used if we don't know the first term:

$$a_n = a_k + (n-k)\cdot d$$

Let's find the general term of the following linear sequences:

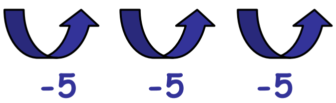
a) 12, 15, 18, 21, 24, ... b) 100, 95, 90, 85, 80, ...

a) 12 , 15 , 18 , 21 , ... $a_1 = 12$ $d = 3$



$$a_n = a_1 + (n-1)\cdot d = 12 + (n-1)\cdot 3 = 12 + 3n - 3 = \underline{\underline{9 + 3n}}$$

b) 100, 95, 90, 85, ... $a_1 = 100$ $d = -5$



$$a_n = a_1 + (n-1)\cdot d = 100 + (n-1)\cdot(-5) = 100 - 5n + 5 = \underline{\underline{105 - 5n}}$$

For a linear sequence, the common difference tells you the multiple of *n* in the general term.

Exercise 8

Find the general term of the following arithmetic sequences:

a) 1, 8, 15, 22, ... b) $-8, -14, -20, -26, \ldots$ c) $-\frac{5}{2}, -2, -\frac{3}{2}, -1, \ldots$

Exercise 9

Find the 86^{th} term of the sequence $-11, -7, -3, 1, \ldots$

Exercise 10

Find the difference in a linear sequence where $a_5 = -5$ and $a_{19} = 65$.

Exercise 11

Find the 60^{th} term of a linear sequence where the fifteenth term is 21 and the difference is 1/3.

Exercise 12

Find the number of terms (n) in a linear sequence where the first term is 7, the last term is 112, and the difference is 3.

Exercise 13

Find the value of the 30^{th} term of an arithmetic sequence if $a_7 = 24$ and $a_{13} = 72$.

Exercise 14

A gardener plants borders with roses and marigolds using the following pattern.

Find a rule for the number of marigolds that will be planted with *n* roses. How many marigolds will be planted with 10 roses?

Exercise 15

Jo has £20 in her piggy bank. In each case, find a rule for the amount of money she will have in the piggy bank after *n* weeks if she saves:

a) £3 a week   b) £5 a week   c) £10 a week

Exercise 16

Caroline has won a prize of 1000 tins of dog food.

In each case, find a formula for the number of tins she will have left after *n* weeks if her dog eats:

a) 5 tins a week   b) 7 tins a week   c) 14 tins a week

Exercise 17

An author has signed a contract to write a book of 400 pages.

In each case, find a formula for the number of pages left to write after n days if the author writes:

a) 17 pages a day b) 20 pages a day c) 25 pages a day

Exercise 18

The first row in a theatre is 4.5 m from the stage, and the eighth row is 9.75 m.

a) What is the distance between two rows?
b) How far is the 17th row from the stage?

Sum of the n first terms of a linear sequence

Let's start solving this problem:

How do you add the numbers from 1 to 5000 without actually doing it or using a calculator?

If you wrote out all the numbers from 1 to 5000 and then wrote them backwards underneath, you would have twice as many numbers as you needed, but the problem is easier, here is why:

1	2	3	4	....	4998	4999	5000
5000	4999	4998	4997	....	3	2	1
5001	5001	5001	5001	....	5001	5001	5001

Notice that if we add the two lists, we get a list that is the same number, 5001, repeating. In fact, since each of the lists is 5000 numbers long, we have, in the sums, a list of 5000 numbers that are each 5001.

You have to admit that adding 5000 5001's is a lot quicker than the other way since $5000 \times 5001 = 25{,}005{,}000$.

But wait, you say, that's too much. We were only supposed to add one list and we added two. Okay, then the answer must be half as much:

$$1+2+3+4+...+5000 = \frac{25{,}005{,}000}{2} = \underline{\underline{12{,}502{,}500}}$$

You can use this method if you have any number of consecutive numbers, whether they start with 1 or not. In fact, the numbers do not have even to be consecutive. They just have to be in an arithmetic sequence.

Let's add up all the odd numbers from 1 to 25. We do the same thing as before:

1	3	5	....	21	23	25
25	23	21	....	5	3	1
26	26	26	....	26	26	26

We have to know how many numbers there are, and in this case there are 13 twenty-sixes, so the total must be $\frac{13\times 26}{2}=169$.

In general, if you have a linear sequence of n numbers and you know the first and last one, you can find the sum, S_n, by:

$$S_n=\frac{(a_1+a_n)\cdot n}{2}$$

Example: Find the sum of the first 30 terms of the sequence

4, 7, 10, 13, 16, …

1. Identify $a_1=4$ and $n=30$.
2. Calculate a_{30} using the general term formula:

$$a_n=a_1+(n-1)d \;\Rightarrow\; a_{30}=4+(30-1)\cdot 3=4+29\cdot 3=91$$

3. Substitute:

$$S_n=\frac{(a_1+a_n)\cdot n}{2} \;\Rightarrow\; S_{30}=\frac{(4+91)\cdot 30}{2}=1425$$

Exercise 19

The first term of a linear sequence is 16, and the tenth term is 43. Calculate a_{20} and S_{20}.

Exercise 20

Calculate the sum of the first ten terms of a linear sequence where $a_{10}=58$ and $d=6$.

Exercise 21

Calculate the number of terms (n) in a linear sequence knowing that the first term is 7, $a_n = 53$ and $S_n = 300$.

Exercise 22

The medicine dose is 100 mg the first day and 5 mg off every following day. The treatment lasts 12 days. How many milligrams must the sick person have during the whole treatment?

Exercise 23

A ball rolling on an inclined plane covers 1 m in the 1st second, 4 m in the 2nd, 7 m in the 3rd, and so on. How many metres does it cover in 20 seconds?

3.3.- GEOMETRIC SEQUENCES

Sequences of numbers that follow a pattern of **multiplying a fixed number from one term to the next** are called **geometric sequences**. This fixed number is called **common ratio**. The following sequences are geometric sequences:

Sequence A: 1, 2, 4, 8, 16, … ratio = 2
Sequence B: 0.01, 0.06, 0.36, 2.16, 12.96, … ratio = 6
Sequence C: 16, -8, 4, -2, … ratio = -1/2

The common ratio can be found dividing one term by the preceding.

General term of a geometric sequence

Let's find the general term of a geometric sequence:

Example: 5, 10, 20, 40, 80, …

The first term in this sequence is 5 and the ratio, **r**, between consecutive terms is 2.

$$a_1 = 5$$
$$a_2 = 5 \times 2$$
$$a_3 = 5 \times 2 \times 2 = 5 \times 2^2$$
$$a_4 = 5 \times 2 \times 2 \times 2 = 5 \times 2^3$$
$$a_5 = 5 \times 2 \times 2 \times 2 \times 2 = 5 \times 2^4$$
………………………………………………
$$a_n = 5 \times 2 \times 2 \times \ldots \times 2 = \underline{\underline{5 \times 2^{n-1}}}$$

The general term of a geometric sequence can be calculated from the first term and the ratio using this expression:

$$a_n = a_1 \cdot r^{n-1}$$

The following expression can also be used if we don't know the first term:

$$a_n = a_p \cdot r^{n-p}$$

Let's find the general term of the following geometric sequences:

a) 2, 6, 18, 54, ... b) 200, 100, 50, 25, ...

a) 2 , 6 , 18 , 54 , ... $a_1 = 2$ $r = 3$

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$$a_n = a_1 \cdot r^{n-1} = \underline{\underline{2 \cdot 3^{n-1}}}$$

(Be careful, powers of different base cannot be multiplied)

b) 200, 100, 50, 25, ... $a_1 = 200$ $r = \frac{1}{2}$

$\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

$$a_n = a_1 \cdot r^{n-1} = \underline{\underline{200 \cdot \left(\frac{1}{2}\right)^{n-1}}}$$

Exercise 24

Find out which of the following sequences are geometric. Find the ratio and the nth term for the geometric ones.

a) 2, 4, 8, 16, 32, ... b) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ c) $\frac{7}{2}, \frac{7}{4}, \frac{7}{8}, \frac{7}{16}, \ldots$

d) $1, \frac{4}{3}, 2, \frac{16}{5}, \ldots$ e) $\sqrt{3}, 3, 3\sqrt{3}, 9, \ldots$

Exercise 25

Find the first four terms of the following geometric sequences:

a) $a_n = 5^n$ b) $a_n = 5 \cdot (-1)^n$ c) $a_n = \frac{160}{2^n}$

Exercise 26

Find the ratio of a geometric sequence where $a_{15} = 192$ and $a_9 = 2187$.

Exercise 27

Find a_7 in a geometric sequence where $a_4 = 125$ and $r = \frac{5}{2}$.

Exercise 28

In a geometric sequence we know that $a_{11} = 256$ and $a_6 = \frac{8}{243}$. Find the value of a_{16} and the general term.

Exercise 29

£3000 is invested at 5% compound interest. Find the principal after 8 years.

Exercise 30

A canning machine costs £20,000. Each year the canning machine depreciates in value by 15%. Work out the value of the canning machine after 5 years.

Sum of the n first terms of a geometric sequence

The sum of the n first terms of a geometric sequence can be expressed by a formula. Let's find it:

The sum of the n first terms of a geometric sequence is:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

Multiplying both sides by r:

$$S_n \cdot r = a_1 \cdot r + a_2 \cdot r + a_3 \cdot r + \ldots + a_{n-1} \cdot r + a_n \cdot r = a_2 + a_3 + a_4 + \ldots + a_n + a_n \cdot r$$

Subtracting:

$$\begin{array}{rl} S_n \cdot r = & \quad a_2 + a_3 + a_4 + \ldots + a_n + a_n \cdot r \\ S_n = & a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \\ \hline S_n \cdot r - S_n = & -a_1 \qquad\qquad\qquad\qquad + a_n \cdot r \end{array}$$

$$S_n \cdot r - S_n = -a_1 + a_n \cdot r \rightarrow S_n \cdot (r-1) = a_n \cdot r - a_1$$

$$S_n = \frac{a_n \cdot r - a_1}{r-1} = \frac{a_1 \cdot r^{n-1} \cdot r - a_1}{r-1} = \frac{a_1 \cdot r^n - a_1}{r-1} = \frac{a_1 (r^n - 1)}{r-1}$$

$$S_n = \frac{a_1(r^n - 1)}{r - 1}, \text{ where } r \neq 1$$

Example: Find the sum of the 12 first terms of the sequence where $a_3 = 8$ and $a_8 = 256$.

1. We calculate the ratio:

$$a_8 = a_3 \cdot r^5 \to 256 = 8 \cdot r^5 \to r^5 = \frac{256}{8} = 32 \to r = \sqrt[5]{32} = 2$$

2. We find the first term:

$$a_3 = a_1 \cdot r^2 \to 8 = a_1 \cdot 2^2 \to a_1 = \frac{8}{2^2} = 2$$

3. We substitute in the expression above:

$$S_{12} = \frac{a_1(r^{12} - 1)}{r - 1} = \frac{2(2^{12} - 1)}{2 - 1} = 8190$$

Exercise 31

Find the sum of the first 7 terms of a geometric sequence where the second term is 300 and the ratio is 1/2.

Exercise 32

The first term of a geometric sequence is 6, the last term is 1458, and the sum is 2184. Find the ratio and the number of terms.

Sum of the terms of a geometric sequence where -1<r<1

Remember the formula of the sum of the first n terms of a geometric sequence:

$$S_n = \frac{a_1(r^n - 1)}{r - 1} = \frac{a_1(1 - r^n)}{1 - r}$$

When the common ratio is a positive number less than 1, $0 < r < 1$, we can check that:

If n approaches $\infty \Rightarrow r^n$ approaches 0

$$\text{Example: If } r = 0.5 \Rightarrow \begin{cases} r^{10} = 0.5^{10} \simeq 0.00098 \\ r^{30} = 0.5^{30} \simeq 0.0000000093 \\ r^{50} = 0.5^{50} \simeq 0.00000000000000089 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

So, the sum of 'all terms' of a geometric sequence where $0 < r < 1$ will be:

$$S_{\infty} = \frac{a_1}{1-r}$$

This formula is also true when $-1 < r < 0$.

This sum is called the **infinite sum**.

Exercise 33

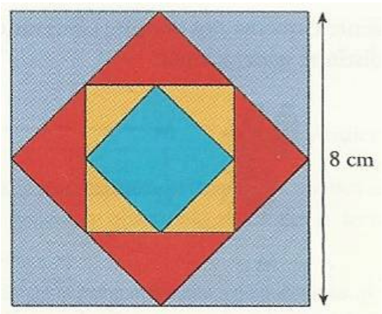
The first term of a geometric sequence is 8 and the common ratio is $r = 0.75$. Find the infinite sum.

Exercise 34

The fourth term of a geometric sequence is 10 and the sixth one is 0.4. Find the common ratio, the first term, the sum of the first 8 terms and the infinite sum.

Exercise 35

Have a look at the different squares in this figure. They appear joining the midpoints of two adjoining sides:



a) Find out the areas of the first six squares of this sequence. What is the general term of the sequence?
b) Write the sequence of the lengths of the sides.
c) Find the sum of the areas of 'all squares' created in this way.